

# On the locus of non-rigid hypersurfaces

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We show that the Zariski closure of the set of hypersurfaces of degree  $M$  in  $\mathbb{P}^M$ , where  $M \geq 5$ , which are either not factorial or not birationally superrigid, is of codimension at least  $\binom{M-3}{2} + 1$  in the parameter space.

Bibliography: 21 titles.

**1. Formulation of the main result and scheme of the proof.** Let  $\mathbb{P}^M$ , where  $M \geq 5$ , be the complex projective space,  $\mathcal{F} = \mathbb{P}(H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M)))$  the space parametrizing hypersurfaces of degree  $M$ . There are Zariski open subsets  $\mathcal{F}_{\text{reg}} \subset \mathcal{F}_{\text{sm}} \subset \mathcal{F}$ , consisting of hypersurfaces, regular in the sense of [14], and smooth, respectively. The well known theorem proven in [14] claims that every regular hypersurface  $V \in \mathcal{F}_{\text{reg}}$  is birationally superrigid. Let  $\mathcal{F}_{\text{srigid}} \subset \mathcal{F}$  be the set of (possibly singular) hypersurfaces that are factorial and birationally superrigid. The aim of this note is to show the following claim.

**Theorem 1.** *The Zariski closure  $\overline{\mathcal{F} \setminus \mathcal{F}_{\text{srigid}}}$  of the complement is of codimension at least  $\binom{M-3}{2} + 1$  in  $\mathcal{F}$ .*

Note that we do not discuss the question of whether  $\mathcal{F}_{\text{srigid}}$  is open or not.

We prove Theorem 1, directly constructing a set in  $\mathcal{F}$ , every point of which corresponds to a factorial and birationally superrigid hypersurface, with the Zariski closure of its complement of codimension at least  $\binom{M-3}{2} + 1$ . More precisely, let  $\mathcal{F}_{\text{qsing} \geq r}$  be the set of hypersurfaces, every point of which is either smooth or a quadratic singularity of rank at least  $r$ . We do *not* assume that singularities are isolated, but it is obvious that for  $V \in \mathcal{F}_{\text{qsing} \geq r}$  the following estimate holds:

$$\text{codim Sing } V \geq r - 1.$$

In particular, by the famous Grothendieck theorem ([7, XI.Cor.3.14], [1]) any  $V \in \mathcal{F}_{\text{qsing} \geq 5}$  is a factorial variety, therefore a Fano variety of index 1:

$$\text{Pic } V = \mathbb{Z}K_V, \quad K_V = -H,$$

where  $H \in \text{Pic } V$  is the class of a hyperplane section.

It is easy to see (Proposition 2) that  $\text{codim}(\mathcal{F} \setminus \mathcal{F}_{\text{qsing} \geq 5}) \geq \binom{M-3}{2} + 1$ .

Denote by  $\mathcal{F}_{\text{reg, qsing} \geq 5} \subset \mathcal{F}_{\text{qsing} \geq 5}$  the subset, consisting of such Fano hypersurfaces  $V \in \mathcal{F}$  that:

- (1) at every smooth point the regularity condition of [14] is satisfied;
- (2) through every singular point there are only finitely many lines on  $V$ .

We obtain Theorem 1 from the following two facts.

**Theorem 2.** *The codimension of the complement of  $\mathcal{F}_{\text{reg, qsing} \geq 5}$  in  $\mathcal{F}$  is at least  $\binom{M-3}{2} + 1$  if  $M \geq 5$ .*

**Theorem 3.** *Every hypersurface  $V \in \mathcal{F}_{\text{reg, qsing} \geq 5}$  is birationally superrigid.*

**Proof of Theorem 2** is straightforward and follows the arguments of [14, 16]; it is given in Section 2.

**Proof of Theorem 3** starts in the usual way [14, 16, 19]: take a mobile linear system  $\Sigma \subset |nH|$  on a hypersurface  $V \in \mathcal{F}_{\text{reg, qsing} \geq 5}$ . Assume that for a generic  $D \in \Sigma$  the pair  $(V, \frac{1}{n}D)$  is not canonical, that is, the system  $\Sigma$  has a *maximal singularity*  $E \subset V^+$ , where  $\varphi: V^+ \rightarrow V$  is a birational morphism,  $V^+$  a smooth projective variety,  $E$  a  $\varphi$ -exceptional divisor and the *Noether-Fano inequality*

$$\text{ord}_E \varphi^* \Sigma > na(E)$$

is satisfied (see [19] for definitions and details). We need to get a contradiction, which would immediately imply birational superrigidity and complete the proof of Theorem 3.

We proceed in the standard way.

Let  $D_1, D_2 \in \Sigma$  be generic divisors and  $Z = (D_1 \circ D_2)$  the self-intersection of the system  $\Sigma$ . Further, let  $B = \varphi(E)$  be the centre of the maximal singularity  $E$ . If  $\text{codim}_V B = 2$ , then

$$\text{codim}_B(B \cap \text{Sing } V) \geq 2,$$

so we can take any curve  $C \subset B$ ,  $C \cap \text{Sing } V = \emptyset$ , and applying [14, Sec.3], conclude that

$$\text{mult}_C \Sigma \leq n.$$

As  $\text{mult}_B \Sigma > n$ , we get a contradiction. So we may assume that  $\text{codim}_V B \geq 3$ .

**Proposition 1 (the  $4n^2$ -inequality).** *The following estimate holds:*

$$\text{mult}_B Z > 4n^2.$$

If  $B \not\subset \text{Sing } V$ , then the  $4n^2$ -inequality is a well known fact going back to the paper on the quartic three-fold [10], so in this case no proof is needed, see [19, Ch. 2] for details. Therefore we assume that  $B \subset \text{Sing } V$ . In that case Proposition 1 is a non-trivial new result, proved below in Sec. 3. The proof makes use of the fact that the condition of having at most quadratic singularities of rank  $\geq r$  is stable with respect to blow ups, in some a bit subtle way. That fact is shown in Sec. 4.

Now we complete the proof of Theorem 3, repeating word for word the arguments of [14]. Namely, we choose an irreducible component  $Y$  of the effective cycle  $Z$ , satisfying the inequality

$$\frac{\text{mult}_o Y}{\deg Y} > \frac{4}{M},$$

where  $o \in B$  is a point of general position. Applying the technique of hypertangent divisors in precisely the same way as it is done in [14] (see also [19, Ch. 3]), we construct a curve  $C \subset Y$ , satisfying the inequality  $\text{mult}_o C > \deg C$ , which is

impossible. It is here that we need the regularity conditions. This contradiction completes the proof of Theorem 3.

**Remark 1.** (i)  $4n^2$ -inequality is not true for a quadratic singularity of rank  $\leq 4$ : the non-degenerate quadratic point of a three fold shows that  $2n^2$  is the best we can achieve.

(ii) Birational superrigidity of Fano hypersurfaces with non-degenerate quadratic singularities was shown in [18]. Birational (super)rigidity of Fano hypersurfaces with isolated singular points of higher multiplicities  $3 \leq m \leq M - 2$  was proved in [17], but the argument is really hard. These two results show that the estimate for the codimension of the non-rigid locus could most probably be considerably sharpened.

(iii) There are a few other papers where various classes of singular Fano varieties were studied from the viewpoint of their birational rigidity. The most popular object was three-dimensional quartics [13, 5, 11, 21]. Other families were investigated in [2, 3]. A family of Fano varieties (Fano double spaces of index one) with a higher dimensional singular locus was recently proven to be birationally superrigid in [12].

(iv) A recent preprint of de Fernex [6] proves birational superrigidity of a class of Fano hypersurfaces of degree  $M$  in  $\mathbb{P}^M$  with not necessarily isolated singularities without assuming regularity. But the dimension of the singularity locus is bounded by  $\frac{1}{2}M - 4$ , and no estimate of the codimension of the complement of this class is given.

## 2. The estimates for the codimension. Let us prove Theorem 2.

First we discuss the regularity conditions in more details. Let  $x$  be a smooth point on a hypersurface  $V$  of degree  $M$  in  $\mathbb{P}^M$ . Choose homogeneous coordinates  $(X_0 : \dots : X_M)$  on  $\mathbb{P}^M$  such that  $x = (1 : 0 : \dots : 0)$ . Then  $V \cap \{X_0 \neq 0\}$  is the vanishing locus of a polynomial

$$q_1 + \dots + q_M$$

where each  $q_i$  is a homogeneous polynomial of degree  $i$  in  $M$  variables  $X_1, \dots, X_M$ . The regularity condition of [14] states that  $q_1, \dots, q_{M-1}$  is a regular sequence in  $\mathbb{C}[x_1, \dots, x_M]$ . In particular,

$$\text{codim}_{\mathbb{A}^M}(\{q_1 = \dots = q_{M-1} = 0\}) = 1.$$

Since all the vanishing loci  $\{q_i = 0\}$  are cones with vertex in  $x$ , the set  $\{q_1 = \dots = q_{M-1} = 0\}$  must consist of a finite number of lines through  $x$ . Hence there also is only a finite number of lines on  $V$  through  $x$ .

If  $x$  is a singular point on  $V$  then  $q_1 \equiv 0$ . The regularity condition (2) is equivalent to

$$\text{codim}_{\mathbb{A}^M}(\{q_2 = \dots = q_M = 0\}) = 1.$$

since  $\{q_2 = \dots = q_M = 0\} \subset V$ , and because of homogeneity every line through  $x$  on  $V$  also lies in  $\{q_i = 0\}$ .

It is not known whether the set  $\mathcal{F}_{\text{reg}}$  is Zariski-open in  $\mathcal{F}$ , but it certainly contains a Zariski-open subset of  $\mathcal{F}$ . The codimension in  $\mathcal{F}$  of its complement  $\mathcal{F} \setminus \mathcal{F}_{\text{reg}}$  is

defined as the codimension of the Zariski closure of the complement. On the other hand,  $\mathcal{F}_{\text{qsing} \geq 5}$  is certainly Zariski-open, hence  $\mathcal{F} \setminus \mathcal{F}_{\text{qsing} \geq 5}$  is Zariski-closed. We have

$$\text{codim}_{\mathcal{F}}(\mathcal{F} \setminus \mathcal{F}_{\text{reg, qsing} \geq 5}) = \min(\text{codim}_{\mathcal{F}}(\mathcal{F} \setminus \mathcal{F}_{\text{reg}}), \text{codim}_{\mathcal{F}}(\mathcal{F} \setminus \mathcal{F}_{\text{qsing} \geq 5})).$$

Hence the estimate of Theorem 2 follows from the following two propositions:

**Proposition 2.** *The codimension of the complement of  $\mathcal{F} \setminus \mathcal{F}_{\text{qsing} \geq 5}$  in  $\mathcal{F}$  is at least  $\binom{M-3}{2} + 1$  if  $M \geq 5$ .*

**Proposition 3.** *The codimension of the (Zariski closure of the) complement of  $\mathcal{F} \setminus \mathcal{F}_{\text{reg}}$  in  $\mathcal{F}$  is at least  $\frac{M(M-5)}{2} + 4$  if  $M \geq 5$ .*

**Proof of Proposition 2.** Let  $S_M := \mathbb{P}^{\binom{M+1}{2}-1}$  be the projectivized space of all symmetric  $M \times M$ -matrices with complex entries. Let  $S_{M,r}$  be the projectivized algebraic subset of  $M \times M$  symmetric matrices of rank  $\leq r$ . The locus  $Q_r(P)$  of hypersurfaces  $H \in \mathcal{F}$  with  $P \in H$  a singularity that is at least a quadratic point of rank at most  $r$  has codimension in  $\mathcal{F}$  equal to

$$\begin{aligned} \text{codim}_{\mathcal{F}} Q_r(P) &= 1 + M + \text{codim}_{S_M} S_{M,r} = 1 + M + \dim S_M - \dim S_{M,r} = \\ &= M + \binom{M+1}{2} - \dim S_{M,r}. \end{aligned}$$

Let  $G(M-r, M)$  be the Grassmann variety parametrizing  $(M-r)$ -dimensional subspaces of  $\mathbb{C}^M$ . To calculate  $\dim S_{M,r}$  we consider the incidence correspondence (see [8, Ex.12.4])

$$\Phi := \{(A, \Lambda) : \Lambda^T \cdot A = A \cdot \Lambda = 0\} \subset S_M \times G(M-r, M).$$

Since the fibers of the natural projection  $\pi_2 : \Phi \rightarrow G(M-r, M)$  is given by a linear subspace of  $S_M$  of dimension  $\binom{r+1}{2} - 1$ , the variety  $\Phi$  is irreducible of

$$\dim \Phi = \binom{r+1}{2} - 1 + r(M-r).$$

Since on the other hand the natural projection  $\pi_1 : \Phi \rightarrow S_M$  is generically 1 : 1 onto  $S_{M,r}$ ,  $\dim \Phi = \dim S_{M,r}$ .

Consequently, since the  $Q_r(P)$  cover  $Q_r$  and  $P$  varies in  $\mathbb{P}^M$ ,

$$\text{codim}_{\mathcal{F}} Q_r \geq \text{codim}_{\mathcal{F}} Q_r(P) - M = \binom{M-r+1}{2} + 1.$$

This completes the proof of Proposition 2.

For  $r = 4$ , we have  $\text{codim}_{\mathcal{F}} Q_4 \geq M$  if

$$\text{codim}_{\mathcal{F}} Q_4 - M \geq \binom{M-3}{2} + 1 - M = \frac{(M-2)(M-7)}{2} \geq 0,$$

hence if  $M \geq 7$ .

**Proof of Proposition 3.** Let  $\Phi = \{(x, H) : x \in H\} \subset \mathbb{P}^M \times \mathcal{F}$  be the incidence variety of hypersurfaces of degree  $M$  in  $\mathbb{P}^M$ . Let  $\Phi_{\text{reg}}$  be the subset of pairs  $(x, H)$  satisfying the regularity conditions. Note that the Zariski closure  $\overline{\Phi \setminus \Phi_{\text{reg}}}$  in  $\Phi$  maps onto the Zariski closure  $\overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}}$  in  $\mathcal{F}$ . The fiber of  $\Phi_{\text{reg}}$  over a point  $x \in \mathbb{P}^M$  under the natural projection  $\pi_1 : \mathbb{P}^M \times \mathcal{F} \rightarrow \mathbb{P}^M$  can be described as

$$\Phi_{\text{reg}}(x) := \{H : x \in H \text{ satisfies the regularity conditions}\} \subset \mathcal{F}.$$

Choosing homogeneous coordinates  $(X_0 : \dots : X_M)$  on  $\mathbb{P}^M$  such that  $x = (1:0:\dots:0)$  we can write  $\mathcal{F} = \mathbb{P}H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M))$  as a projectivized product

$$\mathcal{F} = \mathbb{P}(\bigoplus_{i=0}^M \mathcal{P}_{i,M} \cdot X_0^{M-i}),$$

where the  $\mathcal{P}_{i,M}$  are the vector spaces of homogeneous polynomials in  $X_1, \dots, X_M$  of degree  $i$ . In particular, the  $\pi_1$ -fiber  $\Phi(x)$  of  $\Phi$  over  $x$  is  $\mathbb{P}(\bigoplus_{i=1}^M \mathcal{P}_{i,M} \cdot X_0^{M-i})$ .

For another point  $x' \in \mathbb{P}^M$  also choose homogeneous coordinates  $(X'_0 : \dots : X'_M)$  on  $\mathbb{P}^M$  such that in these new coordinates  $x' = (1 : 0 : \dots : 0)$ . Then the projective-linear automorphism on  $\mathbb{P}^M$  given by the coordinate change from  $(X_0 : \dots : X_M)$  to  $(X'_0 : \dots : X'_M)$  maps a polynomial  $F(X_0, \dots, X_M)$  to the polynomial  $F(X'_0, \dots, X'_M)$ . In particular, the induced linear automorphism on the affine cone  $H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M))$  over  $\mathcal{F}$  maps the product structure  $\prod_{i=0}^M \mathcal{P}_{i,M} \cdot X_0^{M-i}$  onto the product structure  $\prod_{i=0}^M \mathcal{P}'_{i,M} \cdot (X'_0)^{M-i}$ . Hence the induced projective-linear automorphism on  $\mathcal{F}$  maps  $\Phi(x)$  onto  $\Phi(x')$  and  $\Phi_{\text{reg}}(x)$  to  $\Phi_{\text{reg}}(x')$  because the regularity conditions only depend on these product structures.

Consequently, the  $\pi_1$ -fibers of the Zariski closure  $\overline{\Phi \setminus \Phi_{\text{reg}}}$  are the Zariski closure  $\overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)}$ , hence

$$\dim \overline{\Phi \setminus \Phi_{\text{reg}}} = \dim \overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)} + M.$$

Since  $\dim \overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}} \leq \dim \overline{\Phi \setminus \Phi_{\text{reg}}}$  we conclude

$$\begin{aligned} \text{codim}_{\mathcal{F}} \overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}} &\geq \dim \mathcal{F} - \dim \overline{\Phi \setminus \Phi_{\text{reg}}} = \text{codim}_{\mathcal{F}} \overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)} - M = \\ &= \text{codim}_{\Phi(x)} \overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)} - (M - 1). \end{aligned}$$

Let  $\tilde{\Phi}(x) = \prod_{i=1}^M \mathcal{P}_{i,M}$  and  $\tilde{\Phi}_{\text{reg}}(x)$  be the preimages of  $\Phi(x)$ ,  $\Phi_{\text{reg}}(x)$  in the affine cone  $H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(M)) = \prod_{i=0}^M \mathcal{P}_{i,M}$  over  $\mathcal{F}$ . Obviously,

$$\text{codim}_{\Phi(x)} \overline{\Phi(x) \setminus \Phi_{\text{reg}}(x)} = \text{codim}_{\tilde{\Phi}(x)} \overline{\tilde{\Phi}(x) \setminus \tilde{\Phi}_{\text{reg}}(x)}.$$

$\tilde{\Phi}(x) \setminus \tilde{\Phi}_{\text{reg}}(x)$  consists of a subset  $S_1$  Zariski-closed in  $\mathcal{P}_{1,M}^* \times \prod_{i=2}^M \mathcal{P}_{i,M}$  of polynomials  $q_1 + \dots + q_M$  not satisfying regularity condition (1), where  $\mathcal{P}_{1,M}^* = \mathcal{P}_{1,M} \setminus \{0\}$ , and a Zariski-closed subset  $S_2$  of  $\{0\} \times \prod_{i=2}^M \mathcal{P}_{i,M}$  of polynomials  $q_2 + \dots + q_M$  not satisfying

regularity condition (2). Hence  $\overline{\tilde{\Phi}(x) \setminus \tilde{\Phi}_{\text{reg}}(x)}$  is the union of the Zariski closure of  $S_1$  in  $\tilde{\Phi}(x)$  and  $S_2$ . Consequently,

$$\text{codim}_{\tilde{\Phi}(x)} \overline{\tilde{\Phi}(x) \setminus \tilde{\Phi}_{\text{reg}}(x)} = \min(\text{codim}_{\mathcal{P}_{1,M}^* \times \prod_{i=2}^M \mathcal{P}_{i,M}} S_1, \text{codim}_{\tilde{\Phi}(x)} S_2).$$

For  $1 \leq j < i \leq M$  let  $\pi_{i,j} : \mathcal{P}_{1,M}^* \times \prod_{k=2}^i \mathcal{P}_{k,M} \rightarrow \mathcal{P}_{1,M}^* \times \prod_{k=2}^j \mathcal{P}_{k,M}$  be the natural projection. Following the notations in [14] we set for  $k = 2, \dots, M-1$

$$Y_k := \{(q_1, \dots, q_k) \in \mathcal{P}_{1,M}^* \times \prod_{i=2}^k \mathcal{P}_{i,M} : \text{codim}_{\mathbb{P}^M} \{q_1 = \dots = q_k = 0\} < k\},$$

$$R_k := (\mathcal{P}_{1,M}^* \times \prod_{i=2}^k \mathcal{P}_{i,M}) \setminus Y_k,$$

$$\mu_k := \min_{(q_1, \dots, q_{k-1}) \in R_{k-1}} \text{codim}_{\pi_{k,k-1}^{-1}(q_1, \dots, q_{k-1})} (\pi_{k,k-1}^{-1}(q_1, \dots, q_{k-1}) \cap Y_k).$$

$S_1$  can be stratified into disjoint subsets

$$S_1 = \bigcup_{i=2}^{M-1} \pi_{M,i}^{-1}(Y_i) \cap \pi_{M,i-1}^{-1}(R_{i-1}).$$

Each stratum  $\pi_{M,i}^{-1}(Y_i) \cap \pi_{M,i-1}^{-1}(R_{i-1})$  is Zariski-closed in  $\pi_{M,i-1}^{-1}(R_{i-1})$ . Hence

$$\begin{aligned} \text{codim}_{\mathcal{P}_{1,M}^* \times \prod_{i=2}^M \mathcal{P}_{i,M}} S_1 &= \min_{2 \leq i \leq M-1} \text{codim}_{\pi_{M,i-1}^{-1}(R_{i-1})} (\pi_{M,i}^{-1}(Y_i) \cap \pi_{M,i-1}^{-1}(R_{i-1})) \\ &\geq \min_{2 \leq i \leq M-1} \mu_i. \end{aligned}$$

In the same way as for  $S_1$  we obtain

$$\text{codim}_{\prod_{i=2}^M \mathcal{P}_{i,M}} S_2 \geq \min_{2 \leq i \leq M} \nu_i,$$

where

$$\nu_k := \min_{(q_2, \dots, q_{k-1}) \in Q_{k-1}} \text{codim}_{\sigma_{k,k-1}^{-1}(q_2, \dots, q_{k-1})} (\sigma_{k,k-1}^{-1}(q_2, \dots, q_{k-1}) \cap Z_k),$$

$$Q_k := \prod_{i=2}^k \mathcal{P}_{i,M} \setminus Z_k,$$

$$Z_k := \{(q_2, \dots, q_k) \in \prod_{i=2}^k \mathcal{P}_{i,M} : \text{codim}_{\mathbb{P}^M} \{q_2 = \dots = q_k = 0\} < k-1\}$$

and  $\sigma_{k,k-1} : \prod_{i=2}^k \mathcal{P}_{i,M} \rightarrow \prod_{i=2}^{k-1} \mathcal{P}_{i,M}$  is the natural projection. Consequently,

$$\text{codim}_{\prod_{i=1}^M \mathcal{P}_{i,M}} S_2 \geq \min_{2 \leq i \leq M} \nu_i + M,$$

because  $\dim \mathcal{P}_{1,M} = M$ . Using the technique of [14],

$$\mu_i \geq \binom{M}{i}, i = 2, \dots, M-1, \text{ and } \nu_j \geq \binom{M+1}{j}, j = 2, \dots, M.$$

Unfortunately these estimates are too weak for our purposes if  $i = M-1$  and  $j = M$ . Using the technique of [16] we obtain a better estimate for

$$\begin{aligned} \text{codim}_{\pi_{M,M-2}^{-1}(R_{M-2}) \cap \pi_{M,M-1}^{-1}(Y_{M-1})} \pi_{M,M-2}^{-1}(R_{M-2}) \cap \pi_{M,M-1}^{-1}(Y_{M-1}) = \\ \text{codim}_{\pi_{M-1,M-2}^{-1}(R_{M-2}) \cap Y_{M-1}} \pi_{M-1,M-2}^{-1}(R_{M-2}) \cap Y_{M-1}. \end{aligned}$$

First of all,  $\pi_{M-1,M-2}^{-1}(R_{M-2}) \cap Y_{M-1}$  fibers over  $\mathcal{P}_{1,M}^* = R_1$ , hence the codimension is at least the minimal codimension in a fiber. So we can fix a  $q_1 \in R_1$  and choose affine coordinates  $X_1, \dots, X_M$  such that  $q_1 = X_1$ . Restricting the  $q_2, \dots, q_{M-1}$  to  $\{X_1 = 0\} \cong \mathbb{A}^{M-1}$  we obtain homogeneous polynomials in the variables  $X_2, \dots, X_M$ . Hence their vanishing sets can be projectivized in  $\mathbb{P}^{M-2}$ , and setting

$$R'_{M-3} := \{(q_2, \dots, q_{M-2}) : \text{codim}_{\mathbb{P}^{M-2}}(\{q_2 = \dots = q_{M-2} = 0\}) = M-3\} \subset \prod_{i=2}^{M-2} \mathcal{P}'_{i,M-1},$$

$$Y'_{M-2} := \{(q_2, \dots, q_{M-1}) : \text{codim}_{\mathbb{P}^{M-2}}(\{q_2 = \dots = q_{M-1} = 0\}) < M-2\} \subset \prod_{i=2}^{M-1} \mathcal{P}'_{i,M-1}$$

we want to determine a lower bound for

$$\text{codim}_{(\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2}} (\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2}.$$

Here,  $\mathcal{P}'_{i,M-1}$  is the space of homogeneous polynomials of degree  $i$  in  $M-1$  variables  $X_2, \dots, X_M$  and  $\pi'_{M-1,M-2} : \prod_{i=1}^{M-1} \mathcal{P}'_{i,M-1} \rightarrow \prod_{i=1}^{M-2} \mathcal{P}'_{i,M-1}$  is the natural projection.

For each tuple  $(q_2, \dots, q_{M-2}) \in R'_{M-3}$ , integers  $2 \leq b \leq M-2$  and  $2 \leq i_1 < \dots < i_{b-1} \leq M-2$ , there exists a  $b$ -dimensional linear subspace  $L_b \subset \mathbb{P}^{M-2}$  such that  $\{q_{i_1} = \dots = q_{i_{b-1}} = 0\} \cap L_b \subset \mathbb{P}^{M-2}$  has only 1-dimensional components. Vice versa, a tuple  $(q_2, \dots, q_{M-1})$  lies in  $(\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2}$  if for each 1-dimensional irreducible component  $B \subset \{q_2 = \dots = q_{M-2} = 0\}$  spanning the linear subspace  $\langle B \rangle \subset \mathbb{P}^{M-2}$  of dimension  $b$  there exist integers  $2 \leq i_1 < \dots < i_{b-1} \leq M-2$  such that  $\{q_{i_1} = \dots = q_{i_{b-1}} = 0\} \cap \langle B \rangle$  contains  $B$  as a 1-dimensional irreducible component and  $q_{i|B} \equiv 0$  for all  $i \in \{2, \dots, M-1\} \setminus \{i_1, \dots, i_{b-1}\}$  (hence for all  $2 \leq i \leq M-1$ ).

In the terminology of [16]  $q_{i_1}, \dots, q_{i_{b-1}}$  is called a *good sequence* for  $B \subset \langle B \rangle$ . Its existence can be shown inductively, using the regularity condition defining  $R'_{M-3}$ .

If  $b = 1$ , the  $i_1, \dots, i_{b-1}$  do not exist, and the condition restricts to

$$q_{2|B} \equiv \dots \equiv q_{M-1|B} \equiv 0$$

on the line  $B = \langle B \rangle$ .

We can cover  $(\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2}$  by subsets  $Z(b; i_1, \dots, i_{b-1}; L_b)$  consisting of all tuples  $(q_2, \dots, q_{M-1}) \in (\pi'_{M-1,M-2})^{-1}(R'_{M-3})$  such that

$$\dim\{q_{i_1} = \dots = q_{i_{b-1}} = 0\} \cap L_b = 1,$$

$\{q_{i_1} = \dots = q_{i_{b-1}} = 0\} \cap L_b$  contains irreducible components linearly spanning  $L_b$  and  $q_i \equiv 0$  on such a component, for each  $2 \leq i \leq M-1$ . Here,  $1 \leq b \leq M-2$ ,  $2 \leq i_1 < \dots < i_{b-1} \leq M-2$ , and the  $L_b$  are parametrized by the (projective) Grassmann variety  $\mathbb{G}(b, M-2)$  of  $b$ -dimensional linear subspaces  $L_b \subset \mathbb{P}^{M-2}$ . For  $b = 1$ ,

$$Z(1; L) := \{(q_2, \dots, q_{M-1}) : q_i|_L \equiv 0, 2 \leq i \leq M-1\}.$$

All these subsets are Zariski-closed in varying Zariski-open subsets of  $(\pi'_{M-1,M-2})^{-1}(R'_{M-3})$ .

For  $b > 1$  they fiber surjectively onto  $\prod_{k=1}^{b-1} \mathcal{P}'_{i_k}$ . Hence their codimension is estimated by a lower bound for each given  $q_{i_1}, \dots, q_{i_{b-1}}$ , of the codimension of all tuples of  $q_i$ ,  $i \in \{1, \dots, M-1\} \setminus \{i_1, \dots, i_{b-1}\}$ , such that  $q_i|_B \equiv 0$  on an irreducible curve  $B$  linearly spanning  $L_b$ . To find such a lower bound choose homogeneous coordinates  $(X_2 : \dots : X_M)$  such that

$$L_b = \{X_{b+3} = \dots = X_M = 0\}.$$

Then  $q_i \in \mathcal{P}'_{i,M-1}$  cannot vanish on an irreducible curve  $B$  linearly spanning all of  $L_b$  if  $q_i|_{L_b}$  is of the form

$$\prod_{k=1}^i (a_{k,2}X_2 + \dots + a_{k,b+2}X_{b+2}).$$

Consequently the codimension of all  $q_i \in \mathcal{P}'_{i,M-1}$  vanishing on such a curve  $B$  is at least the dimension of the space of polynomials in this form, that is  $b \cdot i + 1$ . Here,  $b$  is the dimension of the space of hyperplanes in  $\mathbb{P}^b$ . It follows that the codimension of  $Z(b; i_1, \dots, i_{b-1}; L_b)$  in (a Zariski-open subset of)  $(\pi'_{M-1,M-2})^{-1}(R'_{M-3})$  is at least

$$\begin{aligned} \sum_{\substack{2 \leq i \leq M-1 \\ i \neq i_1, \dots, i_{b-1}}} (b \cdot i + 1) &\geq b \cdot (2 + \dots + (M-1-b) + (M-1)) + (M-1-b) \\ &= b \cdot \frac{(M-1-b)(M-b)}{2} + (b+1)(M-1) - 2b. \end{aligned}$$

Similarly, the codimension of  $Z(1; L)$  in  $(\pi'_{M-1,M-2})^{-1}(R'_{M-3})$  is at least

$$3 + \dots + M = \frac{M(M+1)}{2} - 3$$

because  $i+1$  is the codimension of the set of polynomials  $q_i \in \mathcal{P}'_{i,M-1}$  vanishing on the line  $L \subset \mathbb{P}^{M-2}$ .

Taking all these data together

$$\text{codim}_{(\pi'_{M-1,M-2})^{-1}(R'_{M-3})} (\pi'_{M-1,M-2})^{-1}(R'_{M-3}) \cap Y'_{M-2}$$



must be at least the minimum of the numbers

$$\begin{aligned} b \cdot \frac{(M-1-b)(M-b)}{2} + (b+1)(M-1) - 2b - (b+1)(M-2-b) \\ = b \cdot \frac{(M-1-b)(M-b)}{2} + b^2 + 1, 2 \leq b \leq M-2, \end{aligned}$$

and

$$\frac{M(M+1)}{2} - 3 - 2(M-3) = \frac{M(M-3)}{2} + 3.$$

Here,  $(b+1)(M-2-b)$  and  $2(M-3)$  are the dimensions of the Grassmann varieties parametrizing the linear subspaces  $L_b$ . An easy analysis of the derivative shows that the function

$$F(b) = b \cdot \frac{(M-1-b)(M-b)}{2} + b^2 + 1$$

is everywhere increasing for  $M \geq 5$ , hence the minimum of  $F(b)$  is  $(M-2)(M-3)+5$  if  $2 \leq b \leq M-2$ . Hence the overall minimum is

$$\frac{M(M-3)}{2} + 3.$$

Following the same line of arguments we also obtain a lower bound for

$$\text{codim}_{\pi_{M,M-1}^{-1}(Q_{M-1})} \pi_{M,M-1}^{-1}(Q_{M-1}) \cap Z_M.$$

First note that it is not necessary to fix  $q_1$  since linear terms do not occur. Hence  $q_2, \dots, q_M$  are polynomials in  $X_1, \dots, X_M$ . Adapting the calculations above shows that the codimension is at least the minimum of the numbers

$$b \cdot \frac{(M-b)(M+1-b)}{2} + b^2 + 1, 2 \leq b \leq M$$

and

$$\frac{(M+1)(M-2)}{2} + 3,$$

that is  $\frac{(M+1)(M-2)}{2} + 3$ , arguing as before.

Finally, all these estimates imply that  $\text{codim}_{\mathcal{F}} \overline{\mathcal{F} \setminus \mathcal{F}_{\text{reg}}}$  is bounded from below by the minimum of the numbers

$$\binom{M}{i} - (M-1), 2 \leq i \leq M-2, \frac{M(M-3)}{2} + 3 - (M-1),$$

$$\binom{M+1}{j} - (M-1) + M, 2 \leq j \leq M, \frac{(M+1)(M-2)}{2} + 3 - (M-1) + M,$$

that is

$$\frac{M(M-3)}{2} + 3 - (M-1) = \frac{M(M-5)}{2} + 4$$

for  $M \geq 5$ .

**3. The  $4n^2$ -inequality.** Let us prove Proposition 1. We fix a mobile linear system  $\Sigma$  on  $V$  and a maximal singularity  $E \subset V^+$  satisfying the Noether-Fano inequality  $\text{ord}_E \varphi^* \Sigma > na(E)$ . We assume the centre  $B = \varphi(E)$  of  $E$  on  $V$  to be maximal, that is,  $B$  is not contained in the centre of another maximal singularity of the system  $\Sigma$ . In other words, the pair  $(V, \frac{1}{n}\Sigma)$  is canonical outside  $B$  in a neighborhood of the generic point of  $B$ .

Further, we assume that  $B \subset \text{Sing } V$  (otherwise the claim is well known), so that  $\text{codim}(B \subset V) \geq 4$ . Let

$$\begin{array}{ccc} \varphi_{i,i-1}: & V_i & \rightarrow V_{i-1} \\ & \cup & \cup \\ & E_i & \rightarrow B_{i-1} \end{array}$$

$i = 1, \dots, K$ , be the *resolution* of  $E$ , that is,  $V_0 = V$ ,  $B_0 = B$ ,  $\varphi_{i,i-1}$  blows up  $B_{i-1} = \text{centre}(E, V_{i-1})$ ,  $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$  the exceptional divisor, and, finally, the divisorial valuations, determined by  $E$  and  $E_K$ , coincide.

As explained in Sec. 4 below, for every  $i = 0, \dots, K-1$  there is a Zariski open subset  $U_i \subset V_i$  such that  $U_i \cap B_i \neq \emptyset$  is smooth and either  $V_i$  is smooth along  $U_i \cap B_i$ , or every point  $p \in U_i \cap B_i$  is a quadratic singularity of  $V_i$  of rank at least 5. In particular, the quasi-projective varieties  $\varphi_{i,i-1}^{-1}(U_{i-1})$ ,  $i = 1, \dots, K$ , are factorial and the exceptional divisor

$$E_i^* = E_i \cap \varphi_{i,i-1}^{-1}(U_{i-1})$$

is either a projective bundle over  $U_{i-1} \cap B_{i-1}$  (in the non-singular case) or a fibration into quadrics of rank  $\geq 5$  over  $U_{i-1} \cap B_{i-1}$  (in the singular case). We may assume that  $U_i \subset \varphi_{i,i-1}^{-1}(U_{i-1})$  for  $i = 1, \dots, K-1$ . The exceptional divisors  $E_i^*$  are all irreducible.

As usual, we break the sequence of blow ups into the *lower* ( $1 \leq i \leq L$ ) and *upper* ( $L+1 \leq i \leq K$ ) parts:  $\text{codim } B_{i-1} \geq 3$  if and only if  $1 \leq i \leq L$ . It may occur that  $L = K$  and the upper part is empty (see [15, 14, 19]). Set

$$L_* = \max\{i = 1, \dots, K \mid \text{mult}_{B_{i-1}} V_{i-1} = 2\}.$$

Obviously,  $L_* \leq L$ . Set also

$$\delta_i = \text{codim } B_{i-1} - 2 \quad \text{for } 1 \leq i \leq L_*$$

and

$$\delta_i = \text{codim } B_{i-1} - 1 \quad \text{for } L_* + 1 \leq i \leq K.$$

We denote strict transforms on  $V_i$  by adding the upper index  $i$ : say,  $\Sigma^i$  means the strict transform of the system  $\Sigma$  on  $V_i$ . Let  $D \in \Sigma$  be a generic divisor. Obviously,

$$D^i|_{U_i} = \varphi_{i,i-1}^*(D^{i-1}|_{U_{i-1}}) - \nu_i E_i^*,$$

where the integer coefficients  $\nu_i = \frac{1}{2} \text{mult}_{B_{i-1}} \Sigma^{i-1}$  for  $i = 1, \dots, L^*$  and  $\nu_i = \text{mult}_{B_{i-1}} \Sigma^{i-1}$  for  $i = L^* + 1, \dots, K$ .

Now the Noether-Fano inequality takes the traditional form

$$\sum_{i=1}^K p_i \nu_i > n \left( \sum_{i=1}^K p_i \delta_i \right), \quad (1)$$

where  $p_i$  is the number of paths from the top vertex  $E_K$  to the vertex  $E_i$  in the oriented graph  $\Gamma$  of the sequence of blow ups  $\varphi_{i,i-1}$ , see [15, 14, 19] for details.

We may assume that  $\nu_1 < \sqrt{2}n$ , otherwise for generic divisors  $D_1, D_2 \in \Sigma$  we have

$$\text{mult}_B(D_1 \circ D_2) \geq 2\nu_1^2 > 4n^2$$

and the  $4n^2$ -inequality is shown. We do not use the following claim, but nevertheless it is worth mentioning.

**Lemma 1.** *The inequality  $\nu_1 > n$  holds.*

**Proof.** Taking a point  $p \in B$  of general position and a generic complete intersection 3-germ  $Y \ni p$ , we reduce to the case of a non log canonical singularity centered at a non-degenerate quadratic point, when the claim is well known, see [4, 20]. Q.E.D.

Obviously, the multiplicities  $\nu_i$  satisfy the inequalities

$$\nu_1 \geq \dots \geq \nu_{L^*} \quad (2)$$

and, if  $K \geq L^* + 1$ , then

$$2\nu_{L^*} \geq \nu_{L^*+1} \geq \dots \geq \nu_K. \quad (3)$$

Now let  $Z = (D_1 \circ D_2)$  be the self-intersection of the mobile system  $\Sigma$  and set  $m_i = \text{mult}_{B_{i-1}} Z^{i-1}$  for  $1 \leq i \leq L$ . Applying the technique of counting multiplicities in word for word the same way as in [15, 14, 19], we obtain the estimate

$$\sum_{i=1}^L p_i m_i \geq 2 \sum_{i=1}^{L^*} p_i \nu_i^2 + \sum_{i=L^*+1}^K p_i \nu_i^2.$$

Denote the right hand side of this inequality by  $q(\nu_1, \dots, \nu_K)$ . We see that

$$\sum_{i=1}^L p_i m_i > \mu,$$

where  $\mu$  is the minimum of the positive definite quadratic form  $q(\nu_*)$  on the compact convex polytope  $\Delta$  defined on the hyperplane

$$\Pi = \left\{ \sum_{i=1}^K p_i \nu_i = n \left( \sum_{i=1}^K p_i \delta_i \right) \right\}$$

by the inequalities (2,3). Let us estimate  $\mu$ .

We use the standard optimization technique in two steps. First, we minimize  $q|_{\Pi}$  separately for the two groups of variables

$$\nu_1, \dots, \nu_{L_*} \quad \text{and} \quad \nu_{L_*+1}, \dots, \nu_K.$$

Easy computations show that the minimum is attained for

$$\nu_1 = \dots = \nu_{L_*} = \theta_1 \quad \text{and} \quad \nu_{L_*+1} = \dots = \nu_K = \theta_2,$$

satisfying the inequality  $2\theta_1 \geq \theta_2$ . Putting

$$\Sigma_* = \sum_{i=1}^{L_*} p_i \quad \text{and} \quad \Sigma^* = \sum_{i=L_*+1}^K p_i,$$

we get the extremal problem

$$\bar{q}(\theta_1, \theta_2) = 2\Sigma_*\theta_1^2 + \Sigma^*\theta_2^2 \rightarrow \min$$

on the ray, defined by the inequality  $2\theta_1 \geq \theta_2$  on the line

$$\Lambda = \left\{ \Sigma_*\theta_1 + \Sigma^*\theta_2 = n \sum_{i=1}^K p_i \delta_i \right\}.$$

Now we make the second step, minimizing  $\bar{q}|_{\Lambda}$ . The minimum is attained for  $\theta_1 = \theta$ ,  $\theta_2 = 2\theta$  (so that the condition  $2\theta_1 \geq \theta_2$  is satisfied and for that reason can be ignored), where  $\theta$  is obtained from the equation of the line  $\Lambda$ :

$$\theta = \frac{n}{\Sigma_* + 2\Sigma^*} \sum_{i=1}^K p_i \delta_i.$$

Now set

$$\Sigma_l = \sum_{i=1}^L p_i, \quad \Sigma_l^* = \sum_{i=L_*+1}^L p_i, \quad \Sigma_u = \sum_{i=L+1}^K p_i$$

(if  $L \geq L_* + 1$ ; otherwise set  $\Sigma_l^* = 0$ ). Obviously, the relations

$$\Sigma_l = \Sigma_* + \Sigma_l^* \quad \text{and} \quad \Sigma^* = \Sigma_l^* + \Sigma_u \tag{4}$$

hold. Recall that, due to our assumptions on the singularities of  $V_i$  we have  $\delta_i \geq 2$  for  $i \leq L$ . Therefore,

$$\theta \geq \frac{2\Sigma_l + \Sigma_u}{\Sigma_* + 2\Sigma^*} n$$

and so

$$\mu \geq 2 \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_* + 2\Sigma^*} n^2.$$

Since

$$\Sigma_l \text{mult}_B Z \geq \sum_{i=1}^L p_i m_i,$$

we finally obtain the estimate

$$\text{mult}_B Z > 2 \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_* + 2\Sigma^*)} n^2.$$

Therefore, the  $4n^2$ -inequality follows from the estimate

$$(2\Sigma_l + \Sigma_u)^2 \geq 2\Sigma_l(\Sigma_* + 2\Sigma^*).$$

Replacing in the right hand side  $\Sigma_* + 2\Sigma^*$  by

$$\Sigma_* + 2(\Sigma_l^* + \Sigma_u) = \Sigma_l + \Sigma_l^* + 2\Sigma_u,$$

we bring the required estimate to the following form:

$$2\Sigma_l^2 + \Sigma_u^2 \geq 2\Sigma_l \Sigma_l^*,$$

which is an obvious inequality. Proof of Proposition 1 is now complete. Q.E.D.

**4. Stability of the quadratic singularities under blow ups.** We start with the following essential

**Definition 1.** Let  $X \subset Y$  be a subvariety of codimension 1 in a smooth quasi-projective complex variety  $Y$  of dimension  $n$ . A point  $P \in X$  is called a quadratic point of rank  $r$  if there are analytic coordinates  $z = (z_1, \dots, z_n)$  of  $Y$  around  $P$  and a quadratic form  $q_2(z)$  of rank  $r$  such that the germ of  $X$  in  $P$  is given by

$$(P \in X) \cong \{q_2(z) + \text{terms of higher degree} = 0\} \subset Y.$$

**Theorem 4.** Let  $X \subset Y$  be a subvariety of codimension 1 in a smooth quasi-projective complex variety  $Y$  of dimension  $n$ , with at most quadratic points of rank  $\geq r$  as singularities. Let  $B \subset X$  be an irreducible subvariety. Then there exists an open set  $U \subset Y$  such that

- (i)  $B \cap U$  is smooth, and
- (ii) the blow up  $\tilde{X}_U$  of  $X \cap U$  along  $B \cap U$  has at most quadratic points of rank  $\geq r$  as singularities.

*Proof.* The statement is obvious if  $B \not\subset \text{Sing}(X)$ . So we assume from now on that  $B \subset \text{Sing}(X)$ .

By restricting to a Zariski-open subset of  $Y$  we may assume that  $B \subset \text{Sing}(X)$  is a smooth subvariety. By assumption there exist analytic coordinates  $z = (z_1, \dots, z_n)$  around each point  $P \in B \subset Y$  such that the germ

$$(P \in X) \cong \{f(z) = z_1^2 + \dots + z_r^2 + \text{terms of higher degree} = 0\} \subset Y.$$

Then the singular locus  $\text{Sing}(X)$  is contained in the vanishing locus of the partial derivatives of this equation, hence in

$$\left\{ \frac{\partial f}{\partial z_1} = \dots = \frac{\partial f}{\partial z_r} = 0 \right\}.$$

Since

$$\frac{\partial f}{\partial z_i} = 2z_i + \text{terms of higher degree}, 1 \leq i \leq r,$$

setting  $z'_1 := \frac{1}{2} \frac{\partial f}{\partial z_1}, \dots, z'_r := \frac{1}{2} \frac{\partial f}{\partial z_r}, z'_i := z_i$  for  $r+1 \leq i \leq n$  yields new analytic coordinates

$$z'_1, \dots, z'_r, z'_{r+1}, \dots, z'_n$$

of  $Y$  around  $P$ . In these new coordinates the defining equation of  $X$  still is of the form

$$z'^2_1 + \dots + z'^2_r + \text{terms of higher degree} = 0,$$

and  $B \subset \{z'_1 = \dots = z'_r = 0\}$ . Perhaps after a further coordinate change we can even assume that

$$B = \{z'_1 = \dots = z'_k = 0\}, k \geq r.$$

*Claim.*  $(P \in X) \cong \{z'^2_1 + \dots + z'^2_r + f_{\geq 3} = 0\}$  where  $f_{\geq 3}$  consists of terms of degree  $\geq 3$  and is an element of  $(z'_1, \dots, z'_k)^2$ .

*Proof of Claim.*  $B \subset \text{Sing}(X)$  must be contained in  $\left\{ \frac{\partial f_{\geq 3}}{\partial z'_j} = 0 \right\}$ , hence  $\frac{\partial f_{\geq 3}}{\partial z'_j} \in (z'_1, \dots, z'_k)$  for all  $k+1 \leq j \leq n$ . This is only possible if  $f_{\geq 3} \in (z'_1, \dots, z'_k)$ . Write  $f_{\geq 3} = z'_1 f'_1 + \dots + z'_k f'_k$ . Then as before  $\frac{\partial f_{\geq 3}}{\partial z'_i} = f'_i + \sum_{1 \leq j \leq k, j \neq i} z'_j \frac{\partial f'_j}{\partial z'_i} \in (z'_1, \dots, z'_k)$  for all  $1 \leq i \leq k$ . But this is only possible if  $f'_i \in (z'_1, \dots, z'_k)$  for all  $1 \leq i \leq k$ .  $\square$

Using the coordinates  $z'_1, \dots, z'_n$  we can cover the blow up of  $Y$  along  $B$  over  $P \in Y$  by  $k$  charts with coordinates

$$t^{(i)}_1, \dots, z_i, \dots, t^{(i)}_k, z_{k+1}, \dots, z_n, 1 \leq i \leq k,$$

where  $z'_j = t^{(i)}_j z_i$  for  $1 \leq j \leq k, j \neq i, z'_i = z_i$  and  $z'_l = z_l$  for  $k+1 \leq l \leq n$ . To prove the theorem we only need to check in each chart that along the fiber of the exceptional divisor over  $P \in B$  there are at most quadratic points of rank  $\geq r$  as singularities. We distinguish several cases:

*Case 1.*  $1 \leq i \leq r$ , say  $i = 1$ .

Then the strict transform of  $X$  is given by the equation

$$1 + (t^{(1)}_2)^2 + \dots + (t^{(1)}_r)^2 + z_1 \cdot F + Q(t^{(1)}_2, \dots, t^{(1)}_k) \cdot G = 0,$$

where  $Q$  is a quadratic polynomial in  $t^{(1)}_2, \dots, t^{(1)}_k$  and  $G \in (z_{k+1}, \dots, z_n)$ . On the fiber of the exceptional divisor over  $P, \{z_1 = z_{k+1} = \dots = z_n = 0\}$ , the gradient of this function can only vanish when  $t^{(1)}_2 = \dots = t^{(1)}_r = 0$ . But this locus does not

intersect the strict transform, hence in this chart the strict transform is smooth along the fiber of the exceptional divisor over  $P$ .

*Case 2.*  $r + 1 \leq i \leq k$ , say  $i = k$ .

Then the strict transform of  $X$  is given by the equation

$$(t_1^{(k)})^2 + \dots + (t_r^{(k)})^2 + z_k \cdot F + Q(t_1^{(k)}, \dots, t_{k-1}^{(k)}) \cdot G = 0,$$

$Q$  and  $G$  as above. On the fiber of the exceptional divisor over  $P$ ,  $\{z_k = z_{k+1} = \dots = z_n = 0\}$ , the gradient of this function can only vanish when  $t_1^{(k)} = \dots = t_r^{(k)} = 0$ . We first discuss the origin in these coordinates,

$$(0, \dots, 0) \in \{t_1^{(k)} = \dots = t_r^{(k)} = z_k = z_{k+1} = \dots = z_n = 0\}.$$

If  $F$  has a constant term then the strict transform of  $X$  is smooth in  $(0, \dots, 0)$ .

If  $F$  has no constant terms but contains linear terms then the rank of the quadratic term in the defining equation is still  $\geq r$  because we only add quadratic monomials containing  $z_k$  to  $(t_1^{(k)})^2 + \dots + (t_r^{(k)})^2$ . Hence  $(0, \dots, 0)$  is a quadratic point of rank  $\geq r$ .

Finally, if  $F$  is of degree  $\geq 2$  the quadratic term in the defining equation is  $(t_1^{(k)})^2 + \dots + (t_r^{(k)})^2$ . Hence  $(0, \dots, 0)$  is a quadratic point of rank  $r$ .

The affine coordinate change to

$$t_1^{(k)}, \dots, t_r^{(k)}, t_{r+1}^{(k)} - a_{r+1}, \dots, t_{k-1}^{(k)} - a_{k-1}, z_k, z_{k+1}, \dots, z_n$$

leads to a defining equation of the strict transform around the point

$$(0, \dots, 0, a_{r+1}, \dots, a_{k-1}, 0, 0, \dots, 0) \in \{t_1^{(k)} = \dots = t_r^{(k)} = z_k = z_{k+1} = \dots = z_n = 0\}$$

in one of the forms already discussed. Consequently, in this chart all points in the strict transform of  $X$  also lying on the fiber of the exceptional divisor over  $P$  are smooth or quadratic points of rank  $\geq r$ .  $\square$

**Remark 2.** Note that  $\tilde{X}_U$  is again a subvariety of codimension 1 in the smooth quasi-projective blow up of  $U$  along  $B \cap U$ . The universal property of blow ups [9, Prop.II.7.14] and the calculations in the proof above tell us that the exceptional locus  $E_U \subset \tilde{X}_U$  is a Cartier divisor on  $\tilde{X}_U$  such that the morphism  $E_U \rightarrow B \cap U$  is a fibration into quadrics of rank  $\geq r$  in a  $\mathbb{P}^{\text{codim}_Y B}$ -bundle.

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